ISSN 2072-0149

The AUST

JOURNAL OF SCIENCE AND TECHNOLOGY

Volume - 7 Issue : 1 & 2
January 2015 & July 2015
(Published in July 2018)

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On The Diophantine Equation $3^x + 31^y = z^2$

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Abstract: In this paper the Diophantine equation $3^x + 31^y = z^2$ has been solved. It is found that the equation has exactly one non-negative integer solution for x, y and z and the solution is (1,0,2).

1. Introduction

Acu (2007) solved the Diophantine equation $2^x + 5^y = z^2$ and he got exactly two solutions for non-negative integers (x,y,z) which are (3,0,3), (2,1,3).

Rabago (2013) solved two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ and found that the equations have exactly two solutions for non-negative integers (x,y,z) which are {(1,0,2), (4,1,10)} and {(1,0,2), (2,1,10)} respectively.

The Diophantine equations $5^x + 31^y = z^2$, $7^x + 29^y = z^2$, $13^x + 23^y = z^2$ have been solved by Rabago (2013). The author found that each equations have only one non-negative integer solution (1,1,6).

Shivangi and Madan (2017) solved the exponential Diophantine equation $3^x + 13^y = z^2$ and found that the equation has exactly four non-negative integer solutions (1,0,2), (1,1,4), (3,2,14) and (5,1,16) in the form of (x,y,z).

Sroysang (2012, 2013) solved the Diophantine equations $3^x + 5^y = z^2$ and $3^x + 17^y = z^2$. The author found that each equations have only one non-negative integer solution (1,0,2).

So far the Diophantine equation $3^x + 31^y = z^2$ has not been solved yet. Thus the equation has been solved in this paper.

2. Preliminaries

Proposition 2.1. (a,b,x,y) = (3,2,2,3) is a unique solution for the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers such that min {a,b,x,y} > 1 (Catalan's conjecture) (Chotchaisthit (2004)).

Lemma 2.2. The Diophantine equation $3^x + 1 = z^2$ has a unique solution (x,z) = (1,2) where x and z are non-negative integers (Sroysang (2012).

Proof: Let x and z be non-negative integers in the equation $3^x + 1 = z^2$.
If z = 0, then $3^x + 1 = 0$ or, $3^x = -1$ which is impossible.
If x = 0, then $z^2 = 2$ is not possible. Then $x \geq 1$.
Thus, $z^2 = 3^x + 1 \geq 3^1 + 1 = 4$. Then $z \geq 2$.
Now, we consider the equation in the form $z^2 - 3^x = 1$ which is similar to the proposition 2.1.
By proposition 2.1, we have x = 1 then z = 2.

Hence (1,2) is a unique solution for the equation $3^x + 1 = z^2$ where (x,z) are non-negative integers.

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Lemma 2.3. The Diophantine equation $1 + 31^y = z^2$ has no non-negative integer solution (y,z).

Proof. Let y and z be non-negative integers in the equation $1 + 31^y = z^2$.
If $y = 0$, then $z^2 = 2$ which is impossible. It follows that $y \geq 1$.
Thus, $z^2 = 1 + 31^y \geq 1 + 31^1 = 32$. We obtain that $z \geq 6$.
Now, we consider this equation in the form $z^2 - 31^y = 1$.
By Proposition 2.1, we obtain that $y = 1$. Thus is a contradiction with the proposition 2.1.
Hence the equation $1 + 31^y = z^2$ has no non-negative integer solution.

3. New Results

Theorem 3.1. (1,0,2) is a unique non-negative integer solution (x,y,z) for the Diophantine equation $3^x + 31^y = z^2$ where x, y and z are non-negative integer.

Proof: Let x,y and z be non-negative integer in $3^x + 31^y = z^2$.
If $x = 0$, then $1 + 31^y = z^2$ has no non-negative integer solution (by Lemma 2.3).
Thus, we have $x \geq 1$. Now we divide the number y into three cases as follows:

Case 1. If $y = 0$. Lemma 2.2 follows that $x = 1$ and $z = 2$.

Case 2. If y is even, say $y = 2n$ where n is a positive integer then

$3^x + 31^{2n} = z^2$

or, $3^x = (z - 31^n)(z + 31^n)$

or, $3^{(x-u)} 3^u = (z - 31^n)(z + 31^n)$ where u is a non-negative integer and $x > 2u$.

Thus, $(z - 31^n) = 3^u$ then $(z + 31^n) = 3^{(x-u)}$

Now, $(z + 31^n) - (z - 31^n) = 3^{(x-u)} - 3^u$

or, $31^n . 2 = 3^u (3^{(x-2u)} - 1)$.

Thus, $u = 0$ and $3^x - 1 = 31^n . 2$

Adding both side by -2, we obtain $2(31^n - 1) = 3(3^{(x-1)} - 1)$.
That is $x = 2$ and $31^n - 1 = 3$ or, $31^n = 4$, which is a contradiction.
Thus, $3^x + 31^y = z^2$ is not possible for even positive integer.

Case 3. When y is odd, say $y = 2k + 1$ where k is a non-negative integer. Then we can write

$3^x + 31^y = z^2$ as $3^x + 31^{2k} . 31 = z^2$

or, $3^x + 31^{2k} . 6 = z^2 - 31^{2k} . 25$

or, $3^x + 31^{2k} . 6 = (z + 31^k . 5)(z - 31^k . 5)$.

Note that $3^x + 31^y = z^2$ has a solution in positive integer then z is even say $z = 2p$ for some natural number p. Then we have,

$3^x + 31^{2k} . 6 = (2p + 31^k . 5)(2p - 31^k . 5)$.

This equation has two possibilities.

$$\begin{cases} (2p - 31^k . 5 = 1 \\ 2p + 31^k . 5 = 3^x + 31^{2k} . 6) \end{cases}$$

or,

$$\begin{cases} (2p + 31^k . 5 = 1 \\ 2p - 31^k . 5 = 3^x + 31^{2k} . 6) \end{cases}$$

Solving the first set of equalities we have,

$31^k (10 - 31^k . 6) = 1(3^x - 1)$ which implies that

$31^k = 31^0$ and $3^x - 1 = 10 - 31^k . 6$

This gives that $k = 0$ and $3^x = 5$. But it is not possible.

Again solving the second set of equalities we get,

$31^k(31^k . 6 + 10) = 1(1 - 3^x)$ which implies that

$31^k = 1$ or, $k = 0$

and $6.1 + 10 = 1-3^x$ or, $3^x = -15$ which is impossible.

Therefore, by case 1, case 2 and case 3, (1,0,2) is only non-negative integer solution of the Diophantine equation $3^x + 31^y = z^2$.

Corollary 3.2 The Diophantine equation $9^x + 31^y = z^2$ has no non-negative integer solution where x, y and z are non-negative integer.

Proof: Suppose that there are non-negative integers x, y and z such that

$9^x + 31^y = z^2$

or, $3^{2x} + 31^y = z^2$.

Let $u = 2x$ then $3^u + 31^y = z^2$.

By theorem 3.1 it follows that $(u,y,z) = (1,0,2)$.

Thus, $u = 1$ then $x = 1/2$ it is a contradiction.

Hence $9^x + 31^y = z^2$ has no non-negative integer solution.

Corollary 3.3 The Diophantine equation $3^x + 31^y = z^4$ has no non-negative integer solution where x, y and z are non-negative integer.

Proof: Suppose that there are non-negative integers x, y and z such that $3^x + 31^y = z^4$.

Let $u = z^2$ then $3^x + 31^y = u^2$.

By theorem 3.1 it follows that $(x,y,u) = (1,0,2)$.

Thus, $u = 2$ then $z^2 = 2$ it is a contradiction.

Hence $3^x + 31^y = z^4$ has no non-negative integer solution.

Corollary 3.4 (1,0,2) is a unique non-negative integer solution (x,y,z) for the Diophantine equation $3^x + 961^y = z^2$ where x, y and z are non-negative integer.

Proof: Suppose that there are non-negative integers x, y and z such that,

$3^x + 961^y = z^2$

or, $3^x + 31^{2y} = z^2$.

Let $u = 2y$ then $3^x + 31^u = z^2$.

By theorem 3.1 it follows that $(x,u,z) = (1,0,2)$.

Thus, $u = 0$ then $y = 0$.

Hence (1,0,2) is only non-negative integer solution of the Diophantine equation $3^x + 961^y = z^2$.

4. Conclusion

In this paper the Diophantine equation $3^x + 31^y = z^2$ and $3^x + 961^y = z^2$ have been solved. It is found that each equations have exactly one non-negative integer solution for x, y and z and the solution is (1,0,2). The Diophantine equation $9^x + 31^y = z^2$ and $3^x + 31^y = z^4$ also have been solved and found that no non-negative integer solution where x, y and z are non-negative integer.

References:

D. Acu, On a Diophantine equation, General Mathematics, 15(4) (2007),145-148.

P. Chotchaisthit, Primary Ccolotomic units and a proof of Catalan's conjecture, J. Reine Angeq. Math., 27 (2004),167-195.

J. F. T. Rabago, On Two Diophantine Equation $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$. International Journal of Mathematics and Scientific Computing, 3(1) (2013), 28-29.

J. F. T. Rabago, More on Diophantine Equation of Type $p^x + q^y = z^2$. International Journal of Mathematics and Scientific Computing, 3(1) (2013), 15-16.

B. Sroysang, On the Diophantine equation $3^x + 5^y = z^2$. International Journal of Pure and Applied Mathematics, 81(4)(2012), 605-608.

B.Sroysang, On the Diophantine equation $3^x + 17^y = z^2$. International Journal of Pure and Applied Mathematics, 89(1) (2013), 111-114.

A. Shivangi, M. S. Madan, On the Diophantine equation $3^x + 13^y = z^2$. International Journal of Pure and Applied Mathematics, 114 (2) (2017), 301-304.